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ON THE MAGNETOHYDROSTATIC THEORY OF SUNSPOTS

Inaugural Dissertation
For the Attaining of the Doctorate Degree

from the

Scientific Faculty

of the

Ludwig-Maximilians-University

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Submitted by

Willi Deinzer of Nuernberg

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Summary

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Stationary sunspot models have been calculated on the basis of the following assumption. The magnetic field of the sunspot is in magneto-hydrostatic equilibrium with its environment and the horizontal component of the pressure gradient required for this is to be created by interfering with convection, the mechanism of energy transport essential in the area of the so-called hydrogen-convection zone, through the magnetic field. Based on the Prandtl theory of mixing length, this interference is phenomenologically described through a ratio of the mixing length l to the scalar height of the pressure H through $0 \leq l/H \leq (l/H)_{max}$. Under corresponding boundary conditions we thus obtain a single-parameter group of models of different effective temperatures and different magnetic-field intensities.

Author

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1 - Introduction.

Sunspots are the dark spots on the surface of the sun discovered by Christoph Scheiner, S.J., in 1611. With increasingly improved techniques of observation, they have proved themselves to be an extraordinarily many-sided astrophysical phenomenon. Their main characteristics are limited duration, the characteristic migration from higher heliographic latitudes to the vicinity of the equator of the sun and the numerical increase of their occurrence during an eleven-year cycle as well as the intense magnetic fields accompanying them and the characteristics of motion of the matter related to them. The following investigation is concerned with stationary sunspots and attempts to give a quantitative explanation for the joint occurrence of areas with reduced reflection of energy ("sunspots") -- in relation to their environment -- and intense magnetic fields. The separate treatment of this partial problem which is independent of time, is suggested by the relatively slow temporal variability of the phenomenon and appears to be expedient in view of the multiplicity of the physical processes inherent in the sunspots.

According to T. G. Cowling (1946), the decay time of the sunspots and magnetic fields predominates over all other time scales occurring in relation with the phenomenon so that we may consider the magnetic field as a primary and the reduced reflection of energy as a secondary factor. We are therefore assuming in the following the existence of an intense magnetic field and shall attempt to explain the thermodynamic state of solar matter related to this.

A magnetic field exerts certain forces on ionized matter, i.e., solar matter in this case, by way of the electric currents flowing in the latter. Stationary conditions exist then when -- in accordance with the laws of magnetohydrostatics -- pressure gradient and gravity form an equilibrium

with these magnetic forces. If we assume symmetry of rotation and freedom of distortion, it will be easily apparent that a magnetic field exerts forces on its environment parallel and perpendicular to the axis of symmetry. If these assumptions are accepted for the sunspot magnetic fields, we may then anticipate forces in a vertical and in a horizontal direction (axis of symmetry of the field in the direction of the radius of the sun). When in equilibrium, the pressure gradient must therefore possess a horizontal component, i.e., the pressure within the range of the magnetic field must be dependent not only on the depth but also on the distance of the axis of symmetry. The curve of pressure and temperature in depth, the so-called stratification, is principally determined by the mechanism which transports the energy created within the interior of the sun to the surface. When in equilibrium, this mechanism of energy transport must be influenced by the magnetic field so that the resulting pressure gradient together with gravity compensates the magnetic forces.

This fact brings magnetohydrostatics which describe the relation between pressure gradient and magnetic field, and the laws which determine stratification and the influence of the magnetic field on the latter, in relation with each other. The formulation of the magnetohydrostatic equilibrium, generally a partial vector-differential equation, can be simplified as shown by A. Schlueter and St. Temesvary (1958). On the basis of appropriate assumptions, it is possible to accomplish integration in a horizontal direction which makes it possible to restrict the considerations to the axis of symmetry and to calculate the magnetic field here by means of a standard differential equation.

Within the range between solar surface and a depth of about 1/10 of the solar radius -- within the range of the so-called hydrogen-convection

zone (WKZ) [Wasserstoff-Konvektions-zone] -- transport of energy takes place almost completely through turbulent convection. Since a deductive description of this state on the basis of the laws of hydrodynamics is not yet developed sufficiently far as that it could be utilized here successfully, the description of the WKZ is made as heretofore on the basis of the Prandtl theory of mixing length: a "turbulent element" forms, rises, completely intermixes with its environment after covering the Prandtl mixing length ℓ, and thus imparts its excess energy to the environment. In astrophysical application, the mixing length ℓ is generally postulated as proportionate to the scalar height of the pressure H (designated as homogeneous layer thickness in the following) by assuming a value between 1 and 2 for the constant of proportionality. This theory is purely phenomenological, i.e., the formulation is selected so that it furnishes a physically reasonable description of the WKZ.

In the determination of the stratification within the range of the magnetic fields, we are then concerned with investigating the influence of the magnetic field on turbulent convection. For this case also, a deductive description has not yet been sufficiently tested so that it is expedient to employ here also a phenomenological theory. It thus becomes useful to follow in more detail the influence of the ratio of mixing length ℓ to homogeneous layer thickness H on the stratification of the WKZ. This ratio controls the effectiveness of the convective energy transport. If it is zero, convection in relation to energy transport is completely ineffective and energy is transported entirely through radiation with the pressure and/or temperature curves behaving correspondingly. With an increase of ratio, the share of convection in energy transport also increases until it corresponds to the customary concept at a value of $\ell/H = (\ell/H)_{WKZ}$

and we obtain the customary WKZ stratification.

The influence of the magnetic field on turbulent convection must be assumed qualitatively in a similar manner. The higher the intensity of the magnetic field, the more greatly will it interfere with the motion of matter related to convection and thus reduce the share of convection in energy transport until it becomes so high in the border case that convection is completely suppressed and a stratification corresponding to energy transport by radiation is brought about.

L. Biermann suggested in 1941 that the sunspot phenomenon be explained on the basis of this border case of stratification corresponding to energy transport by radiation within the range of intense magnetic fields. Because quantitative execution on the basis of complete suppression of convection by the magnetic fields initially lead to difficulties, an attempt was made to take into account an interference with convection. As a measure for this interference, we have available the ratio of mixing length l to homogeneous layer thickness H in view of the preceding discussion. In the following, we therefore describe the physical mechanism of the interference with convection by a magnetic field with the help of the number $0 \leq l/H \leq (l/H)_{WKZ}$.

The calculations performed show that this formulation is successful. The theoretical sunspot models, as solutions of the system of differential equations describing the sunspots under physically logical boundary conditions, are realistic, i.e., their orders of magnitude agrees with the observations as far as they can be compared to the latter (e.g., in regard to magnetic-field intensities). In accordance with the varying degree of interference with convection by magnetic fields of varying extent which are in equilibrium with stratifications characterized by varying degrees of effective spot temperatures, we obtain a system of models. We may conclude

from this that the physical concept of the relation between reduced energy flow and magnetic field is accurate in principle. Indications on the effectiveness of the interference with convection by a magnetic field also follow from the discussion of the system of sunspot models.

Table of Symbols

$\mathfrak{H}$ = Vector of magnetic-field intensity; its components in the system of cylindrical coordinates are called B_r, B_φ, B_z; B_o = Intensity in center of field at the observable level.

o_1, o_2 = Integration constants, cf. equations (22) and (23).

o_p = Specific heat at constant pressure.

D = Auxiliary magnitude required for the conversion and synthesis of the system of equations in Chapter 26 for deriving the convective temperature gradient.

$D(\alpha)$ = Normed horizontal distribution of B_z.

F = Total flow of energy through the sunspot composed of radiation energy flow F_R and convective energy flow F_K.

f = Magnetic flux in special normalization, cf. equation (5).

$\mathfrak{g}$ = Vector of gravity acceleration at the direction parallel to the solar radius at the amount g.

H = Homogeneous layer thickness, cf. equation (10).

k = Boltzmann constant.

l = Prandtl mixing length.

$P(\alpha)$ = Horizontal distribution of B_z.

p = Gas pressure.

r = Distance from the axis of symmetry of the sunspot.

$\bar{r}$ = Mean radius of spot, definition cf. equation (8).

$\mathfrak{R}$ = Gas constant.

T = Absolute temperature; T_{eff} = effective temperature.

U = Auxiliary magnitude, cf. D.

V = Auxiliary magnitude, cf. D.

$\bar{v}$ = Mean velocity of "turbulent elements".

W = Auxiliary magnitude, cf. D.

X = Mass component of hydrogen.

x = Degree of ionization of hydrogen.

y = Extent of Magnetic field defined in connection with equation (5).

z = Depth considered positive downward; special depths, z_D = depth of upper boundary (so-called geometric depression) and z_o = depth of lower boundary.

α = Auxiliary magnitude for the description of the magnetic field

ξ by means of the law of similitude, cf. equation (2).

$\varkappa$ = Coefficient of absorption.

μ = Molecular weight, μ_o = value at degree of ionization $x = o$.

ϱ = Density.

σ = Radiation constant in the Stefan-Boltzmann law.

τ = Optical depth.

Φ = Magnetic flux.

φ = Angle around the axis of symmetry of the sunspot (azimuth).

χ = Ionization energy of hydrogen.

∇ = Double-logarithmic temperature gradient $\frac{d \log T}{d \log p}$; special gradients, ∇_{ad} adiabatic, ∇_r = radiation gradient, and ∇' gradient describing the change of state of the rising "turbulent elements".

2 - Differential Equations.

The qualitative concepts described earlier are now to be investigated in greater detail which makes it necessary to formulate them quantitatively. The general framework for such a formulation is constituted by the balance of impulse and energy derived from the Boltzmann equation of collision. By adaptation of the individual members to the present case, we obtain, as expression for the conservation of impulse, the magnetohydrostatic equation -- employed in the following for the determination of the magnetic field -- and, as expression for the conservation of energy, the equations of stratification. This remark is intended only to sketch the general physical background of the present investigation but the procedure leading to the specific equations will not be followed and the latter indicated directly in their form familiar from literature.

a) Magnetic Field.

The differential equation to be indicated here will be found, together with its derivation, in the communication of A. Schlueter and St. Temesvary (1958). The manner in which it is obtained will be reviewed briefly here and some of the matters inherent in this connection are discussed. In regard to details, cf. the original communication.

The equilibrium between magnetic forces and pressures is described by the magnetohydrostatic equation

$$\operatorname{grad} p = -\frac{1}{4\pi}[\mathfrak{B}, \operatorname{rot}\mathfrak{B}] + \varrho\mathfrak{k}. \qquad (1)$$

As appropriate system of coordinates for the treatment of this equation, we have selected the system of cylindrical coordinates: r = distance from the symmetry axis, z = depth considered positive downward, and φ = angle around the axis. In order to simplify the member due to the Lorentz force, the following assumptions have been made: the magnetic field is meridional ($B_\varphi = 0$) and axially symmetric (independent of φ). The second assumption

results in the fact that one of the three equations (1) is identically satisfied in the selected coordinate system. A considerable simplification is produced by the further assumption that the magnetic flux is similar at all depths -- expressed mathematically as

$$B_z(z,r) = B_z(z,0) \cdot P(\alpha) \quad \text{, with} \quad \alpha(z,r) = r \cdot \xi(z). \tag{2}$$

This assumption is meaningful only if it is applied to a magnetic field in which no retrogressive field lines occur, i.e., the total magnetic flux Φ is finite. That sunspot magnetic fields satisfy this additional postulate follows from the observations of G. E. Hale and S. B. Nicholson (1925) according to which two sunspots of opposed magnetic polarity are always paired. The function $P(\alpha)$ describes the horizontal dependence of the vertical component B_z of the magnetic field. In accordance with the additional postulate, the latter must always be positive, decrease adequately toward infinite and -- consistent with equation (2) -- assume the value 1 at the point $\alpha = 0$.

It is now possible to express the components of the magnetic field by $P(\alpha)$ and $\xi(z)$. From the constancy of magnetic flux together with equation (2), the vertical component follows as

$$B_z(z,r) = D(\alpha) \cdot \xi'(z); \tag{3}$$

the function $D(\alpha)$ follows from $P(\alpha)$ through appropriate normalization as

$$D(\alpha) = \frac{\Phi}{2\pi \int_0^\infty P(\alpha)\,\alpha\,d\alpha} P(\alpha).$$

From the freedom of divergence of the magnetic field $\operatorname{div} \mathfrak{B} = 0$ postulated by the Maxwell equations together with equation (3), we further obtain for the horizontal component

$$B_r(z,r) = -D(\alpha)\,\alpha\,\frac{d\xi}{dz}. \tag{4}$$

The expressions (3) and (4) are inserted in the r-component of equation

(1) so that we obtain an equation which can be integrated by way of α. After rearrangement, there results

$$f y y'' - y^4 + 8\pi \Delta p = 0 \qquad (5)$$

In this, strokes signify derivations from the depth z, i.e.,

$$y = \sqrt{B_z(z,0)}, \qquad \Delta p = p(z,\infty) - p(z,0)$$

$$f = \frac{\phi}{\pi} \frac{\int_0^\infty P^2(\alpha)\,\alpha\,d\alpha}{\int_0^\infty P(\alpha)\,\alpha\,d\alpha}, \qquad = p_{Wkz}(z) - p_{Fleck}(z).$$

This very simple equation can be interpreted as the equilibrium between magnetic pressure $\frac{y^4}{8\pi}$, magnetic intensities $\frac{f y y'}{8\pi}$ and horizontal pressure differential Δ_p. Made possible by the law of similitude, integration by way of the horizontal dependence of the magnitudes shows that it is possible to reduce a partial differential equation to a standard differential equation and simultaneously offers the advantage of being able to restrict all considerations to the axis of symmetry.

When applied to the z-component of equation (1), this results in

$$p' = g \cdot \varrho . \qquad (6)$$

because the Lorentz force vanishes on the axis of the spot. The vertical magnetohydrostatic equilibrium is consequently reduced to the hydrostatic equilibrium and the equation therefore already belongs to the second part of the system.

As shown in A. Schlueter and St. Temesvary (1958), the equation of the field lines reads

$$\alpha(z,r) = \alpha = \text{const.}$$

and there results from this, in accordance with equation (2),

$$r = \frac{\alpha}{\xi(z)}$$

The function $\xi(z)$ occurring herein can be expressed by means of equation (3) through B_z and $D(\alpha)$, i.e., in the form for r = 0 which is useful subsequently. If we now take into account also the normalization which

connects $P(\alpha)$ with $D(\alpha)$, we then obtain as equation of the field lines

$$r = \sqrt{\frac{\phi}{\pi}\frac{1}{B_z(z,r)}}\;\frac{\alpha}{\sqrt{2\int_0^{\alpha} P(\alpha)\,\alpha\,d\alpha}} \tag{7}$$

This formula is written so that it can be brought into relation with the definition of the mean spot radius $\bar{r}$

$$\pi \bar{r}^2 \cdot B_z(z,0) = \phi \qquad \bar{r} = \sqrt{\frac{\phi}{\pi}\frac{1}{B_z(z,0)}} \tag{8}$$

The mean spot radius so defined therefore is the field line characterized by

$$\alpha = \sqrt{2\int_0^{\bar{\alpha}} P(\alpha)\,\alpha\,d\alpha}$$

For the subsequent calculation, we now still have to select the function $P(\alpha)$ which shall be

$$P(\alpha) = e^{-\alpha^2} \tag{9}$$

This formula agrees well with the distribution of magnetic flux observed on the solar surface, (cf. J. C. de Jager, (1959), fig. 33), possesses the required properties, and the integrals occurring in the various expressions among which it is included [?] can be carried out. We then obtain

$$\begin{aligned} B(\alpha) &= \frac{\phi}{\pi}\cdot e^{-\alpha^2}, \\ f &= \frac{\phi}{\pi}\cdot\frac{1}{2}, \\ \bar{r} &= \alpha\sqrt{\frac{\phi}{\pi}}\,\frac{1}{y}. \end{aligned} \tag{7'}$$

According to definition (8), the mean spot radius $\bar{r}$ is therefore the field line belonging to $\alpha = 1$, i.e., it lies where the vertical component B_z assumes 1/e of the value of the field-center intensity.

b) Stratification.

The stratification of the sun in the range between surface and a depth of about 1/10 of solar radius of interest here is determined by two standard differential equations of the first order. Especially suitable for

numerical execution are the equations in a form with z and log T as dependent variables and log p as independent variable.

For the depth z, there follows from equation (6) the differential equation

$$\frac{dz}{d\log p} = \ln 10 \cdot \frac{\partial \pi}{\mu g} = \ln 10 \cdot H \cdot \tag{10}$$

If energy transport takes place through radiation, there then results for log T the differential equation

$$\frac{d\log T}{d\log p} = \nabla_R = \frac{3}{16} \frac{\kappa \cdot \varrho \cdot H}{\sigma T^4} F. \tag{11}$$

If the consequent temperature gradient exceeds the adiabatic gradient

$$\nabla_{ad} = \frac{1 + \varkappa(1-\varkappa)(S/2 + \chi/c_p T)}{S + \varkappa(1-\varkappa)(S/2 + \chi/c_p T)^2}, \tag{12}$$

stratification then becomes thermally instable. In the case of the then resulting turbulent convection, the description of stratification is then affected with the problematics described earlier. On the basis of the Prandtl theory of mixing length, a formalism was developed for the calculation of the temperature gradient and the generally known form of the latter can be found in E. Boehm-Vitense (1958). The total flow of energy is composed of a (principal) component F_K which is transported by convection, and a (generally smaller by some powers of ten) component F_R which is transported by radiation, i.e.,

$$F = F_K + F_R . \tag{13}$$

Both components are linked with the temperature gradient ∇ and the gradient ∇' determining the temperature variation of the rising turbulent elements through the equations

$$\begin{aligned} F_R &= \frac{16}{3} \cdot \frac{\sigma T^4}{\kappa \varrho H} \nabla , \\ F_K &= \frac{1}{2} c_p \varrho T \bar{v} \frac{l}{H} (\nabla - \nabla') . \end{aligned} \tag{14}$$

The velocity $\bar{v}$ of the rising "turbulent elements" results as the equilibrium between buoyancy and Newton's resistance and amounts to

$$\bar{v}^2 = \frac{1}{2} \frac{g}{\mu} \left(\frac{l}{H}\right)^2 (\nabla - \nabla') . \tag{15}$$

If the "turbulent elements" did not exchange energy with their environment during their existence, their change of state would take place adiabatically; if they reflect energy to their environment, their change of temperature results according to E. Boehm-Vitense (1958) from

$$\frac{\nabla - \nabla'}{\nabla' - \nabla_{ad}} = \frac{1}{24} \frac{c_p \varrho^2 \bar{T} \varkappa H \bar{v}}{\sigma \bar{T}^4} \left(\frac{l}{H}\right) . \tag{16}$$

These equations permit us to calculate the temperature gradient in a case of thermal instability. In regard to questions on the exact physical background and the derivation, reference is made here also to the original communication which includes the bibliography of the entire pertinent literature. For subsequent calculation, it will be found convenient to combine the equations suitably. According to R. Kippenhahn (1960), this is done as follows: From equations (15) and (16) we obtain an equation which links $\nabla - \nabla'$ with the so-called excess gradient $\nabla - \nabla_{ad}$,

$$\begin{aligned} &(\nabla - \nabla') + 2u (\nabla - \nabla')^{1/2} - (\nabla - \nabla_{ad}) = 0 , \\ &u = 12 \frac{\sigma \bar{T}^4}{c_p \varrho^2 \bar{T} \varkappa H} \left(\frac{H}{l}\right)^2 \left(\frac{\partial \mu}{\partial \bar{T}}\right)^{1/2} . \end{aligned} \tag{17}$$

From the equations (11), (13), (14), and (15), we obtain an equation for $\nabla - \nabla'$ which can be converted, by means of the positive root of (17), into the following equation for $\nabla - \nabla_{ad}$:

$$\begin{aligned} &(\nabla - \nabla_{ad}) - (\nabla_R - \nabla_{ad}) + V \left[\sqrt{u^2 + (\nabla - \nabla_{ad})} - u\right]^3 = 0 , \\ &V = \frac{9}{2} \frac{1}{u} \approx \frac{1}{\Gamma} . \end{aligned} \tag{18}$$

This is an equation of the third order for the magnitude $\sqrt{u^2 - (\nabla - \nabla_{ad})}$. Its solution leads to the so-called "casus reducibilis" and therefore can be represented by the Cardan formulas; from the single real root, there results

with

$$\frac{d\log T}{d\log p} = \nabla = \nabla_{ad} + \frac{1}{W^2}\left(W^2 - \frac{7}{4}u^2 + \frac{2}{3}Wu\right)^2 - u^2,$$
$$W = \sqrt[3]{\frac{u^2}{4}\left(\nabla_R - \nabla_{ad} + \frac{16}{24}u^2\right)^2 + \sqrt{\vartheta}},$$
$$\vartheta = \frac{u^2}{4}\left(\nabla_R - \nabla_{ad} + \frac{16}{24}u^2\right)^2 + \left(\frac{5}{24}u^2\right)^3. \qquad (19)$$

The right-hand sides of the differential equations (10) and (11) and/or (19) are functions exclusively of pressure and temperature. In order to be able to calculate them as such, the functional dependence for some of the magnitudes on these sides needs still to be indicated.

The values of the Rosselland mean of opacity in dependence on pressure and temperature were derived from the communication of E. Vitense (1951a). As with N. Baker and R. Kippenhahn (1962), the value required in each case was obtained through interpolation in a two-dimensional table.

The following relations for c_r, ρ, μ and x are taken from the communication of R. Kippenhahn, St. Temesvary and L. Biermann (1958). The expression for the specific heat c_p reads:

$$c_p = \frac{R}{\mu}\left(\frac{5}{2} + \frac{1}{2}x(1-x)\left(\frac{5}{2} + \frac{\chi}{kT}\right)^2\right).$$

The dependence on density ρ is obtained from the equation of state for ideal gases

$$\rho = \frac{\mu p}{RT}.$$

The molecular weight under partial ionization of the hydrogen results as

$$\mu = \frac{\mu_0}{1+x}.$$

For the degree of ionization x of hydrogen -- only ionization of hydrogen is considered in the following calculations --, we find in the communication cited the form, following out of the Saha equation

$$\log\frac{x^2}{1-x^2} = \frac{5}{2}\log T - \frac{13.55\cdot 5040}{T} - 0.48 - \log x - \log p - \log\mu_0.$$

In these equations, the component of radiation pressure in total pressure is not taken into account since it is negligible in relation to the gas pressure in the range of the sun here under consideration. Moreover,

we calculate with a constant amount g of gravitational acceleration and since the principal part of the solar mass lies below the WKZ, this is also justified.

This places us in possession of a formalism allowing us to calculate the stratification in the range of interest to us. The WKZ stratification was determined in general for $\ell/H = 1$. In order to check the extent to which the results are dependent on ℓ/H, calculation was carried out in one case also for $\ell/H = 2$.

c) Hydrogen-Convection Zone and Magnetic Field.

The formalism constituted in the preceding section is intended to make possible both calculation of the WKZ stratification by taking into account the interfering influence of the magnetic field on convection through reduction of the ratio ℓ/H as well as the determination of stratification on the axis of the spot. The conditions determining the value of ℓ/H will be stated in the next chapter.

This procedure for treating the problem of the sunspots is well adapted to the relationship between magnetohydrostatics and the laws determining stratification mentioned earlier.

The equations of stratification produced a pressure curve which changes within the considered range by some powers of ten. For the calculation of the magnetic field, we need the difference between pressure in the WKZ and pressure along the axis of the spot. As shown by preliminary calculations, this pressure differential may not vary in the same degree as the pressures themselves if we desire to obtain solutions free of singularity of the non-linear equation of the magnetic field. Accordingly, the stratification along the axis of the spot must always closely correspond to the WKZ stratification. Precisely this can be obtained with the indicated description because the two stratifications differ in it only in regard to the value of ℓ/H.

In the attempt to treat the problem under the assumption that the magnetic field completely suppresses convection, it was believed that the lateral irradiation of energy from the hotter environment provides in general for a favorable adaption of the stratifications along the axis of the spot and in the WKZ. However, it was shown that such irradiation is effective only when the mean radius of the spot is smaller by about 1 power of ten than is indicated by the observations. Calculations for spots with the average observed effective temperatures therefore did not produce realistic models (W. Deinzer, 1960).

In order to illustrate the relationship of the selected formulation with the suggestion of Biermann (1941), let us now perform the boundary transition $\ell/H \to 0$ with the equations of stratification. We should note for this purpose that the form for the magnitude U indicated in equation (17) is valid only when the transport of energy takes place through optically thick turbulent elements. According to E. Vitense (1953), its form valid for optically thin "turbulent elements" reads

$$U = 12 \frac{\sigma \pi r \kappa H}{c_p \pi} \left(\frac{g_H}{2\pi}\right)^{1/2},$$

and thus remains constant during boundary transition beginning at the point at which the turbulent elements must be regarded as optically thin. However, the connection indicated in equation (18) for the magnitude V with the magnitudes U valid for optically thick "turbulent elements" is maintained and V therefore becomes zero at the boundary. The boundary transition can therefore be accomplished with equation (18) and we obtain

$$\lim_{\gamma_H \to 0} \nabla = \nabla_R ,$$

which was to be expected also for physical reasons and we obtain in the boundary the stratification corresponding to transport of energy by radiation.

The two following remarks concern the assumption of the constancy of l/H and of the total energy flow F in the range of depth of the spot here under consideration.

Consideration of a dependence on depth of the ratio l/H is in any event negligible in view of the uncertainty contained in the description of convection and interference with it through a magnetic field on the basis of the Prandtl theory of mixing length. From numerical experience, we may even state that a constant value of l/H is physically logical. It will seem evident that the interfering influence of the magnetic field acts at the ratio of magnetic to gas pressure $\mathfrak{H}^2/8\pi p$. This ratio does not change essentially in the upper range close to the surface because -- as shown by the calculations -- field intensity and gas pressure toward the interior increase to the same extent. In the lower ranges where the change of the gas pressure predominates over that of the magnetic pressure, stratification takes place nearly adiabatically. The temperature gradient ∇ then reacts very little to a change of l/H as will be seen from an examination of the differential equation for log T. The latter can be appreciably simplified in the case of almost adiabatic stratification because nearly all energy is transported by convection and the change of state of the rising turbulent elements also takes place almost adiabatically. The temperature gradient ∇ then results from

$$F \approx \tfrac{1}{2} c_p \varrho T \left(\frac{8RT}{\mu}\right)^{1/2} \left(\frac{l}{H}\right)^2 (\nabla - \nabla_{ad})^{3/2} \qquad (20)$$

Under the circumstances indicated, this equation is obtained from equations (14b) and (15). A change of l/H then affects the differential $\nabla - \nabla_{ad}$ in accordance with

$$\frac{\Delta(l/H)}{l/H} = -\frac{3}{4} \frac{\Delta(\nabla - \nabla_{ad})}{\nabla - \nabla_{ad}} \qquad (21)$$

If this difference becomes sufficiently small -- it quickly drops below the value 10^{-2} and lies in the lower ranges at 10^{-5} --, a change of $\mathcal{L}/H$ is then negligible for it.

However, what is the situation in regard to the flow of energy F passing through the area of the spot?*

This question can also be answered accurately only after a deductive description of the turbulence in the magnetic field exists. For the time being, we must attempt here also to work with an assumption. Such an assumption is represented by the suggestion of F. Hoyle (1949) that the convection follows the magnetic field lines. For a magnetic field decreasing in the upward direction, it therefore results that the energy flowing through a certain cross-section is distributed, in its upward movement, over a larger cross-section, i.e., the flow of energy F decreases in the upward direction. This raises the question to what extent an energy flow dependent on depth in this manner influences the calculation of stratification.

Strictly speaking, the "dilution" of the energy through convection along the field lines is due only to the component which is transported by convection. We may consider that convection, immediately after starting, practically assumes the entire transport of energy. For the fraction of the total energy flow transported by convection, there is valid

$$\frac{F_K}{F} \approx 1 - \frac{\nabla}{\nabla_R} .$$

This equation will be found in St. Temesvary (1959) and can be obtained from equations (11), (13), and (14a). Convection begins at the point where the

* A brief clarification of terminology is in order here: The magnitude F, called flow of energy, has the dimension erg sec^{-1} cm^{-2} and should therefore be designated more accurately as density of energy flow. A flow of energy would be the magnitude F integrated over the surface and called luminosity in astrophysics. Since such a parameter does not occur in the following, the customary manner of designation was retained. Specifically, total flow of energy therefore signifies the sum of the convective and the radiation component of F.

radiation gradient ∇_R exceeds the adiabatic gradient ∇_{ad} because of increasing opacity. The increase of opacity in the zone close to the surface takes place very quickly and ∇_R exceeds the value of ∇ at the same rate so that we are justified in the assumption that convection completely assumes the transport of energy immediately after it has begun. This would then mean that the total energy flow F is subject to the "effect of dilution".

An earlier attempt to treat the problem of the sunspots consisted in ascribing the difference spot and WKZ stratification to the "effect of dilution" or, so-to-speak, to the focusing action of the magnetic field lines in regard to the flow of energy. This attempt was unsuccessful since the influence of a total energy dependent on depth was too small; we obtained basically the stratification such as would occur in a sun with the effective temperature of the spots. Moreover, no magnetic fields free of singularity resulted from this. Only a change of ℓ/H produced the decisive advance.

This behavior must be understood as follows: through reduction of the energy flow (corresponding to the effective spot temperatures), we obtain stratifications which differ too greatly from that of the environment; the temperature gradient remains too flat and the pressure increase therefore takes place too steeply. Even an increase of the flow of energy with depth brings about no change in this. The reason for this can be found in the equation

$$\frac{\Delta F}{F} = \frac{3}{\ell} \frac{\Delta(\nabla - \nabla_{ad})}{\nabla - \nabla_{ad}},$$

which follows from equation (20) similar to equation (21); a change of F becomes effective only in the non-adiabatic, i.e., in a spatially relatively closely confined area of the stratification. The change of F over this

range taken under consideration apparently remains too small as that it could effect a more favorable adaptation. Steeper temperature gradients and consequent flatter pressure increase are obtained only if convection is made less effective through reducing the ratio ℓ/H. As will be seen from equation (21), convection is of influence precisely there where the insufficient flow of energy has a particular action.

The reduced flow of energy is therefore confronted by the parameter ℓ/H which is constant over the entire range. We are therefore justified in here neglecting the dependence on depth of F as an effect of higher order. Let us assume therefore a constant flow of energy through the area of the spot which is linked to the effective spot temperature by the Stefan-Boltzmann law in the same manner as the energy flowing through the undisturbed WKZ to the effective temperature of the photosphere.

3 - Boundary Conditions and Calculation Procedure.

Before we can solve the differential equations formulated in the preceding chapter, the latter must be determined more accurately through indication of the boundary conditions. This establishes not only the degrees of freedom of the system of solutions but the kind and position of the boundary conditions essentially influence the manner of the calculation procedure. It therefore seems obvious to follow up discussion of the boundary conditions by an explanation of the calculation procedure.

In the following discussion of the boundary conditions, and for the sake of better illustration, the geometric depth z is considered throughout as an integration variable. The utilization of log p as integration variable mentioned earlier is important only for numerical execution. If we therefore employ the term boundary in the following, this is always done in the meaning of an end of an interval on the geometric depth scale.

Let us turn first to the "upper boundary".

This is to lie respectively in the observable level, i.e., there where the energy radiation mainly takes place. It therefore lies in the layer with the optic depth $\tau = 2/3$. The optical depth is by definition exclusively a function of pressure and temperature. The determination of $\tau = 2/3$ therefore requires the existence of a functional relation between pressure and temperature in this layer. This relation results numerically through integration of the hydrostatic equation $\frac{dp}{d\tau} = \frac{g}{\kappa}$ if the temperature curve and the absorption properties in the overlying atmosphere are known. To a given temperature at the upper boundary, i.e., a given effective temperature, we thus obtain the corresponding pressure. This method should also produce accurate values for the center of sunspots since the Lorenz force vanishes on the axis of symmetry of the magnetic field as mentioned already earlier.

The calculation sketched was carried out by E. Vitense (1951b) by utilizing theoretical temperature stratifications in the atmosphere and theoretical opacities. The findings -- photospheric pressure in dependence on effective temperature and gravitational acceleration -- have been combined in fig. 6 of their communication (cf. A. Unsoeld, 1955, fig. 91). The following calculations of the pressures corresponding to the given effective temperatures were derived from this diagram. However, we can also utilize empirical pressure values if any such values of adequate accuracy are available. We will then find, however, that a variation in the pressure values is compensated by a fluctuation of the so-called geometric depression -- which will be discussed below -- without essentially influencing the further results.

The stratifications were now determined as follows: all boundary conditions required for the calculation of the WKZ were stipulated at the upper boundary, i.e., in the geometric level $z = o$, the effective temperature of the sun and the pressure corresponding to it as found from the diagram mentioned are to be assumed from the stratification. The zero point of the geometric depth scale may be arbitrarily located in the photosphere because the optical depth is not dependent on z. This establishes the zero point for all further considerations.

If we were to proceed in the same manner for the determination of the spot stratification, it would then again be necessary to postulate a number triple z, T, p. There would be no problem in indicating values for pressure and temperature but what should be the value to be given to z? If we were interested exclusively in the spot stratification, we could state here also $z = o$. However, in the further investigation, the sunspot magnetic field is to be calculated specifically from the pressure curve between

spot and environment or, more accurately, from the horizontal pressure differences. It is therefore necessary to reference the geometric depth scales of both stratifications to each other. The boundary value z would therefore have to be equal to the zero-point difference of the depth scales, the so-called geometric depression z_D. The zero-point difference influences the solutions very appreciably and is found qualitatively from the observations but can be determined numerically only extremely inaccurately. The boundary conditions required from the spot stratification were therefore given for the lower boundary.

We thus obtain the values in the observable level as the values of the solutions at the point at which the relation $\tau(p, T) = 2/3$ exists between pressure and temperature. For a given value of l/H, there consequently result given values z_D, T_{eff}, p in the upper boundary, i.e., there exists an unique relation between each of these parameters and l/H. Since we are interested in the entire system of solutions belonging to $o \leqslant /H \leqslant (l/H)_{WKZ}$, it would be entirely in order to presuppose a value for l/H and to perform the calculation with this. However, it is more practical for numerical execution to assume instead T_{eff} -- and thus p -- as existing and to determine l/H and the geometric depression z_D by calculation.

The upper boundary condition to be required from the magnetic field is obtained from the postulate that it and its derivation shall vanish in the infinite. For this there is first one condition necessary: we can safely set the horizontal pressure differentials above the observable levels as equal to zero; equation (5) then reads

$$\underline{\hat{f}yy'' - y^2 = o.}$$

This equation can be integrated once and we obtain

$$\underline{\tfrac{1}{2}\hat{f}y'^2 - \tfrac{1}{6}y^3 = c_1.}$$

If x and y' are to vanish at one point, i.e., in the infinite, then the integration constant c_1 must vanish, i.e., there must be valid

$$f y'' = \frac{1}{2} y^2. \tag{22}$$

That this relation is simultaneously adequate for the behavior in the infinite of the magnetic field required above results from further integration. We obtain, as solution of the differential equation (22)

$$y = \pm \left(\frac{1}{\sqrt{2f}} \cdot \frac{1}{z} + c_2 \right). \tag{23}$$

If $c_1 = o$ and/or equation (22) are satisfied, there are then possible only solutions y(z) which tend, together with their derivations, toward zero in the infinite. From the form (23) of the solutions, we also see the reason why a system of conditions which the magnetic field is to satisfy in the infinite, can be replaced by a single boundary condition. The system of solutions possesses a common asymptote -- y[?]o [illegible] -- which has the required properties. The double sign of the system of solutions (23) is of no significance because the magnetic field of interest here is equal to y^2.

Before we indicate the lower boundary conditions, let us make a short remark on their problematics. In addition to the reason already discussed in the boundary conditions for the stratification, it is primarily the following fact which makes the conditions at the lower boundary necessary: because of the non-linearity of the differential-equation system, the singularities cannot be grasped very simply. In order to avoid solutions during numerical treatment which become singular in the integration interval under consideration (which is easily possible as shown by experience), it is necessary to establish them through conditions at both ends of interval. This self-evident necessity of lower boundary conditions is confronted by

the difficulty of having to make here definite statements in regard to which there do not as yet exist any definite physical concepts. It may therefore be a reasonable compromise to be guided in the formulation of these conditions by a concept arising out of a kind of physics not yet established in details but restricting the occurrence of singular solutions sufficiently in spite of this. One concept of this type is the assumption that the magnetic field undergoes steady transition into a homogeneous magnetic field at the lower boundary. Much more than for merely giving an indication on the behavior of the magnetic field in the interior, this postulate is intended to serve for delimiting the range of validity of the physical concept at the basis of the present theory against a field in which concepts on the creation of the magnetic fields may become essential.

A homogeneous, i.e., a magnetic field y = constant independent of depth, requires pressure differentials Δp = constant independent of depth in accordance with equation (5). Two boundary conditions therefore read

$$y' = 0, \tag{24}$$

$$(\Delta p)' = 0; \tag{25}$$

and a third is contained in the interrelation of the constant values y and Δp. The latter follows from equation (5) as

$$y^* = 8\pi \Delta p. \tag{26}$$

As will be seen, the postulate of a homogeneous magnetic field signifies, due to Δp = constant, an assumption of the findings which can be furnished only by the integration of the stratification equations. The assumption is justified only then when it is postulated at a depth $z = z_o$ at which Δp = constant actually begins to be a solution of the stratification equations. In this case, the solutions are no longer dependent on z_o. If we therefore calculate the system of solutions corresponding to different z_o, we can then see on the behavoir shown by them with increasing z_o whether and when we were justified in postulating the lower boundary conditions. In

point of fact, the solutions converge with increasing z_o. This clarifies the question as to the position of the lower boundary.

There now remains conversion of the condition (25). According to the hydrostatic equation, this is equivalent to $\Delta\varrho = 0$; so that there follows, from the equation of state (on the assumption of degree of ionization $x = 1$ and consequent constant molecular weight μ in these ranges)

$$\frac{\Delta T}{T} = \frac{\Delta p}{p} \tag{27}$$

With the lower boundary conditions (26) and (27) to be required of the stratification, we have now established also the relation of the depth scales of spot and WKZ layers to each other. This is formally expressed in the fact that the latter are a function of horizontal pressure and temperature differences Δp and ΔT.

For the sake of greater clarity, let us here group the boundary conditions which the sunspot models must satisfy:

	Stratification	Magnetic field
above	$\frac{p}{T}$	$y' = \frac{y^4}{\sqrt{2f}}$
below	$\Delta p = \frac{\nu^4}{\varepsilon\pi}$	$y' = 0$
$z = z_o$	$\Delta T = \frac{T}{p} \cdot \Delta p$	

Let us mention further that the position of the upper boundary is determined only through the calculation so that the postulate that the spot stratification shall assume given values of pressure and temperature at the upper boundary, represents only one single condition.

If we now count the degrees of freedom of the system of differential equations and compare them with the number of boundary conditions, we then find that we must expect a 3-parameter system of solutions. The three system parameters are the depth z_o, the magnetic flux Φ and the effective temperature T_{eff} of the spot. From the scheme of the boundary conditions, we

see that the thermodynamic properties of the spot -- characterized by the effective temperature -- are assumed and that the magnetic properties are given and result from the calculations.

The distribution of the boundary conditions becomes clearer if we now explain the calculation procedure. The integration of the system of differential equations was carried out according to the method of Adams and Stoermer. If the difference between interpolated and extrapolated value lay above an upper margin of error, the step-width was halved until this difference was below the margin of error. All calculations were carried out on the G-3 electronic computer of the Max-Planck Institute for Physics and Astrophysics.

Prior to the actual calculation of sunspot models, pressure and temperature curves in the WKZ were calculated. With the condition that this stratification shall assume the empirical photospheric pressure and temperature values in the geometric level $z = o$, the zero point of the geometric depth scale is established and the integration of the stratification equations is determined.

The sunspot model is uniquely determined through the three parameters

$$z_o, \phi, T_{eff}$$

The values of these three parameters were assumed. Moreover, there was also assumed a value for the pressure differential at the lower boundary $\Delta p(z_o)$ so that it then became possible to calculate the value of ΔT from equation (27). The stratification of the WKZ is known so that, with Δp and ΔT, the values of pressure and temperature at the lower boundary are also given and the integration of the stratification can be performed. In order to satisfy now the upper boundary condition -- given pressure and temperature values shall be assumed at the same geometric level -- the relation of mixing length l to homogeneous layer thickness H was changed. It is of advantage

here to utilize a method which varies ζ/H iteratively until the upper boundary condition is satisfied. With the solution so obtained, we have now determined the upper boundary (the so-called geometric depression).

It is now possible to form horizontal pressure differences $\Delta p(z)$ and thus to integrate the equation of the magnetic field. Since postulates are set for y' in both boundaries, it is indifferent at which boundary we begin. We began at the top and satisfied the condition in the lower boundary through variation of the corresponding value of y at the top where an iteration method also offered advantages.

The solution so obtained now allows us to check to what extent equation (26) between the initially assumed value of $\Delta p(z_o)$ and $y(z_o)$ is satisfied. This will generally not be the case. We will therefore derive a new pressure difference $\Delta p(z_o)$ from this equation and repeat the method described iteratively until equation (26) has been satisfied sufficiently accurately. Here also we employed a condensed procedure.

Only such separate integration of stratification and magnetic field and their iterative interlinking makes it possible to avoid extensive numerical difficulties in the treatment of the problem and to prepare the latter for a solution.

4 - Findings.

As repeatedly stated, the solutions of the differential-equation system are uniquely determined under the boundary conditions listed if we presuppose:

a) The depth z_o of the lower boundary;

b) The thermodynamic state of the spot at the upper boundary characterized by the effective spot temperature T_{eff} when utilizing the results of E. Vitense (1951, b) alone;

c) The total magnetic flux Φ through the spot.

We may therefore expect a three-parameter system of solutions. However, only the magnitudes T_{eff} and Φ are physically true system parameters. The magnitude z_o is of significance only in regard to the question of when the solutions will become independent from the depth of the lower boundary. As already mentioned, the concept of a steady transition into a homogeneous magnetic field at the lower border and the consequent lower boundary conditions are justified only then. In order to arrive at an understanding of the respective behavior of the solutions, it is of advantage to examine a representation in which the characteristic properties of the solutions in relation to z_o and in dependence on the parameters T_{eff} and Φ have been plotted.

It is the objective of the present investigation to clarify the connection between the magnetic field and the thermodynamic stratification of sunspots. The mathematical formulation has here been selected so that the thermodynamic state of the spot -- characterized by its effective temperature -- is presupposed and the magnetic field follows out of the calculation. It therefore appears obvious to examine the magnetic field and the functions closely interrelated to it in greater detail for selecting the

properties characterizing the solutions. In Figs. 1, 2-a and 2-b there has been plotted the curve of pressure differential and magnetic field y (z) and/or mean spot radius $\bar{r}$ (z) (cf. para. (8) for definition) as they result for the different total depth z_o. The effective spot temperature $T_{eff} = 4{,}000^o$ and the magnetic flux $\Phi = 5 \times 10^{21}$ Gauss cm^2 have been selected so that they can be considered as representative for the observational cross-section. We see from the figures that the pressure differential tends toward a course uniformly increasing from top to bottom. The magnetic field increases in all cases uniformly from top to bottom and the mean spot radius decreases uniformly from top to bottom. It will therefore be sufficient to regard in greater detail only the value of these functions at the upper and lower boundary in investigating the dependence of the solutions on z_o. Moreover, because of the close interrelation between pressure differential and magnetic field (differential equation (5)), it will be sufficient to pursue further only the behaviour of the magnetic field.

Figs. 3-a and 3-b represent the system corresponding to the parameter T_{eff} and Fig. 4, the system corresponding to the parameter Φ. The values of the respective parameter retained are again those of the observational mean. If we examine the figures, we will note a clear convergence in both systems within a depth of 10,000 km. This takes place more quickly for the values at the upper boundary than those at the lower boundary. This may be due to the fact that the pressure differential is greater at the lower boundary and small variations react more strongly on the magnetic field than at the upper boundary. In any event, this behaviour shows that the magnetic field is not very sensitively dependent on the lower boundary conditions in the observable level. In view of the uncertainty mentioned

in the postulating of lower boundary conditions and since the main attention in all findings is concentrated on the magnitudes which can be compared with observation, this behavior is very satisfactory.

If we examine the figures in detail, we will note that the field intensities which belong to different magnetic flux are not essentially different from each other which is quite the contrary of the system belonging to the effective temperature. The reason for this is based on the very small influence which the change of magnetic flux under consideration exerts on equation (5) and consequently on the calculation of the magnetic field. For the relative variation of y'' which is conditioned by a fluctuation of the magnetic flux, there follows, from equation (5),

$$\frac{\Delta y''}{y''} = - \frac{\Delta \Phi}{\Phi} .$$

Since the magnetic flux varies by somewhat more than a factor of 10 over the entire observed range of sunspots, the relative fluctuation of y'' follows as about 10^{-20}[?]. This property is highly noteworthy because it tends to show that the magnetic field is determined essentially only by the thermodynamic state of the spot.

The dependence of the solutions on the system parameter Φ was investigated only for higher values. For lower values of the magnetic flux, numerical difficulties occurred due to the structure of equation (5). Because of non-linearity of this equation, singular and non-singular solutions lie very close together. In order to obtain a solution free of singularity, the initial values must therefore be indicated accurately within several (physically no longer logical) decimal places during numerical integration. This is extremely time-consuming for values of $\Phi \leqslant 5 \times 10^{21}$ Gauss x cm^2. Moreover, this difficulty is of a purely mathematical nature so that there is no reason in physics not to obtain the results for this

field by extrapolation.

Let us now turn to the system of solutions belonging to T_{eff} and thus to the discussion of the astrophysical findings of the present investigation. As starting point for this, the properties of the solutions under consideration have been grouped in Table 1. The respective values have been derived from Figs. 3-a and 3-b for adequately large values of z_o. The ratio of ℓ/H -- measure for the interference with convection by the magnetic field -- and the depth z_D of the upper boundary of the spot stratification, the so-called geometric depression, change their value almost not at all with increasing total depth and were derived directly from the results of calculation. The flow of energy F follows out of the effective spot temperature T_{eff} from the Stefan-Boltzmann law. The gas pressure in the observable level of the spot is also determined by the effective spot temperature according to E. Vitense (1951-b).

Table 1

T_{eff}	log p	z_D	F	l/H	$y(z_D)$	$y(\infty)$
3800°	5.17	$8.15\ 10^7$	$1.170\ 10^{10}$	0.185	62.0	98
4000°	5.15	$7.00\ 10^7$	$1.425\ 10^{10}$	0.226	58.5	93
4200°	5.12	$5.90\ 10^7$	$1.750\ 10^{10}$	0.277	55.5	87
4400°	5.08	$4.80\ 10^7$	$2.100\ 10^{10}$	0.345	51.5	82
5000°	4.98	$2.40\ 10^7$	$3.500\ 10^{10}$	0.556	42.6	68

We first derive the interrelation between effective spot temperature and the magnetic field-center intensity $B_c = y^2$ of the spot from this table. This constitutes one of the principal objectives of the investigation and has been plotted in Fig. 5. The values of the magnetic field intensities in the observable level are on an order of magnitude of some 1,000 Gauss. High magnetic field intensities correspond to low effective spot temperatures and high effective temperatures (located more closely to the photosphere) are accompanied by low magnetic field intensities. This is physically quite understandable since low effective temperatures characterize

a stratification which is more greatly differentiated from that of the environment than a stratification belonging to high effective temperatures. At lower effective temperatures, there consequently result greater horizontal pressure differentials and consequently also higher magnetic field intensities than vice versa.

The interrelation shown in Fig. 5 offers the possibility of comparison with observations. A connection between effective spot temperature and surface of the umbra, as shown by recent measurements, will be found in P. Stumpff (1961). If we utilize further the empirical interrelation between magnetic field-center intensity and surface of the umbra (cf. J. Houtgast and A. van Sluiters, 1948 or S. B. Nicholson, 1933), we can then compare the findings of observation and of calculation with each other. Plotted in Fig. 5, the measurements lie somewhat below the theoretical curve but show the same course. The approximation of the present theory therefore appears to furnish systematically field intensities somewhat too high but comparison by order of magnitude remains entirely positive.

It is possible further to illustrate the question on the interference with convection by a magnetic field with the aid of the results in Table 1. In Fig. 6, the magnetic-field magnitude $y\ (z_D)$ at the upper boundary of the spot stratification has been plotted against the ratio of mixing length ℓ to homogenuous layer thickness H. It will be seen from this representation that weak magnetic fields are accompanied by values of ℓ/H closer to 1, the value assumed for ℓ/H in the undisturbed WKZ, whereas smaller values of the ratio ℓ/H accompany magnetic fields of higher intensity. This means that intense magnetic fields reduce the component of the flow of energy transported by convection more than weaker magnetic fields which is entirely consistent with the physical concept. Of interest in

this diagram are the border cases $\ell/H = 0$ and $\ell/H = 1$. The resulting sunspot models are restricted on the one hand by the fact that the magnetic field completely suppresses convection. This border case of energy transport exclusively by radiation determines the maximum possible horizontal pressure differentials and thus the maximum possible magnetic field intensities. More intense magnetic fields cannot be a solution of the differential equations treated because they would no longer be in magneto-hydrostatic equilibrium with their environment. Extrapolation of the curve plotted to the value $\ell/H = 0$ produces field intensities of about 5,000 Gauss in the observable level. The other border case belongs to magnetic fields which are so weak that they do not at all influence convection. Even less intense magnetic fields cannot exist because no horizontal pressure differentials are possible lower than those occurring in the border $\ell/H = 1$ and they would also not be in magneto-hydrostatic equilibrium with their environment so that they therefore cannot be solutions of the differential equations. Extrapolation in Fig. 6 to the value $\ell/H = 1$ furnishes an observable field intensity of an order of magnitude of 500 Gauss. Both border cases occur in the observations. If the calculated maximum value of 5,000 Gauss for the magnetic field intensity lies about 1,000 Gauss above the observed value, then the fact that maximum and minimum solutions occur in the calculation, may be in itself and indication of the usefulness of the selected formulation of the problem.

The border case $\ell/H = 0$ throws some light on the models investigated earlier which were based on the assumption that the magnetic field suppresses convection entirely. As already mentioned, lateral irradiation of energy had to provide, in the cases investigated, for keeping the horizontal pressure differentials within the required limits (because of the

singularity-free capability of solution of equation (5)) but resulted in spot radii which were too small (W. Deiner, 1960). The border case $\ell/H = 0$ occurring in the present models should therefore be -- with some discretion -- also a solution of the earlier experiment. Since the radius of the spot does not play any role in the present case, it would appear to follow that the close linkage to the spot radius for the earlier models in the border case of lower effective spot temperature and/or high field intensities is eliminated.

The findings in Table 1 finally permit us to discuss again the question of how the flow of energy in the spot as against the environment can become reduced. As already explained, F. Hoyle (1949) suggested that convection follows the field lines diverging in the upward direction and that this distributes the energy, flowing in from below through a small cross-section, over a larger cross-section above. The flow of energy therefore decreases from bottom to top. As already mentioned, this effect was not taken into consideration in the calculations performed here because it was shown that a reduction of the ratio ℓ/H is far more effective than this effect. Instead of calculating with a total energy flow F dependent on the depth z, the spot stratification was determined immediately on the assumption of uniform distribution everywhere of the energy flow $F = \sigma T_{eff}^4$ in which T_{eff} signifies the effective spot temperature. The present findings now make it possible to estimate <u>a posteriori</u> whether we would actually be in position with Hoyle's suggestion to explain the diminution of the flow of energy. The Hoyle concept of a dilution of the flow of energy can be formulated as

$$\pi \cdot \bar{r}^2 \cdot F = \text{const.}$$

Here we average out over the respective spot cross-section and a large

value of the flow of energy F belongs to a small mean spot radius $\bar{r}$ and vice versa. Together with the definition given in o of the mean spot radius $\bar{r}$, there follows from this formulation

$$\frac{F_{above}}{F_{below}} = \left(\frac{y_{above}}{y_{below}}\right)^2$$

for the corresponding ratio of the magnetic-field magnitudes belonging to the ratio of the flow of energy at different depths [?]. If we insert here the corresponding values for y from Table 1, there then results in all cases the relation F_{above}/F_{below} as about 0.4. This signifies that the Hoyle mechanism alone is generally not able to reduce the entire flow of energy through the higher layers of the sun to the value which is radiated from the sunspot at the top, even if we disregard that various values of F which belong to the various effective spot temperatures do not find any expression at all. We must therefore assume an additional mechanism by which energy is transported from the spot to the environment. Regardless of the manner in which this mechanism may function, there is probably diverted as much more energy from the area of the spot as the magnetic field existing there is more intense. This concept can now also be checked on the basis of the present findings. If we form the difference between the value F_o of the energy flow flowing through the higher undisturbed layers of the sun, and the value F which is transported through the area of the spot under consideration here, we will then actually note that this difference becomes greater with increasing magnetic field.

With this we terminate the discussion of the main astrophysical findings of the calculations performed. In conclusion, we follow this up by the evaluation of two uncertainties which are contained in the present formulation. This is first the question to what extent the pressure value inherent in the upper boundary condition influences the findings. It is

questionable whether it is permissible to utilize here -- as we did -- the findings of E. Vitense (1951, b) in order to obtain the pressure values related to the effective spot temperatures. Observations seem to counter-indicate this. We therefore calculate a sunspot model with the empirical upper boundary values $T_{eff} = 4{,}000^{\circ}$ and log p = 4.25.* A further appreciable uncertainty is contained in the description of turbulent convection and the interference with the latter by a magnetic field on the basis of the Prandtl theory of mixing length.

For the estimation of the latter, it is customary in astrophysics to calculate stratifications for $\ell/H = 1$ and $\ell/H = 2$. From the deviation of the results, we can determine how sensitively these are dependent on ℓ/H. This procedure was also employed in the present case, i.e., the stratification of the WKZ not influenced by magnetic fields was calculated for both cases. Since the further calculations are closely related to this, this causes a change of the entire results. The magnitude of the change may serve as a measure for the sensitivity of effect of this uncertainty. In order to afford satisfactory possibilities for comparison, such a model was also calculated for $T_{eff} = 4{,}000^{\circ}$.

Table 2

log p	$(l/H)_{WKZ}$	z_D	l/H	$y(z_D)$	$y(\infty)$
5.15	1	$7.00\ 10^7$	0.226	58.5	93.0
4.25	1	$4.70\ 10^7$	0.238	56.5	91.0
5.15	2	$3.70\ 10^7$	0.404	53.5	97.5

The findings are grouped in Table 2 in which the findings for $T_{eff} = 4{,}000^{\circ}$ from Table 1 are listed again. Surprisingly enough, the change of pressure at the upper boundary of about 1 power of ten is effective merely in the geometrical depression z_D. Moreover, this takes place in a

*I am indebted for these values to Dr. H. Elsaesser, Goettingen.

favorable sense since the magnitude of the geometrical depression, increased to rather considerable values by employing the findings of Vitense for models of lower effective temperatures, here is brought down to values of a few 100 km which are also suggested by observations.

When examining the other uncertainty, notable changes can also be again found in the value z_D of the geometrical depression and of the value ℓ/H which results for the spot stratification. For the former, there holds good what has already been said in the preceding paragraph. The new value for ℓ/H suggests that we should regard the expression $(\ell/H)/(\ell/H)_{WKZ}$ as a measure for the interference with convection by the magnetic field which is invariant in regard to the stratification of the undisturbed WKZ.

No appreciable change can be noted in the magnitude of the magnetic field although we should not overlook the tendency to somewhat lower values -- which appears to be indicated in both cases -- in view of the systematically somewhat too high values in Table 1.

5 - Conclusion.

The present communication is intended to give an explanation for the interrelation between sunspots and their magnetic field. The magnetic field is to be assumed in magneto-hydrostatic equilibrium with its environment where the pressure gradient required for this in the horizontal direction is assumed to be created through interference with convection by the magnetic field. The interference with convection was taken into account by a reduction of the ratio of Prandtl mixing length ℓ to homogeneous layer thickness H. This formulism permits us to calculate theoretical sunspot models and thus to test the above concept quantitatively.

We basically obtain a single-parameter system of models (the influence of the magnetic flux Φ remains minor) and the system parameter is the ratio ℓ/H. A magnetic field interfering with convection to the indicated extent corresponds on the one hand to a given value of this ratio and determines on the other hand the stratification which is uniquely characterized by the effective temperature. The calculations performed furnish values which agree in order of magnitude with the observations. Since the value of the system parameter must lie in the restrictive interval $0 \leqslant \ell/H \leqslant (\ell/H)_{WKZ}$, the sunspot models are therefore also restricted in the upward and downward direction which also agrees with observations.

Calculations were carried out only so far as was necessary for the determination of the essential physical consequences of the present formulation. Detailed statements -- entirely possible with the formalism developed above -- would seem reasonable only in connection with more accurate observations. A further development of the problem here treated appears to be necessary in three directions:

1) To the extent in which concepts on the origin of the sunspot-magnetic

fields assume quantitative form, better "lower boundary conditions" will result;

2) Further concepts of a mechanism through which energy is transported out of the area of the spot, must be developed and integrated in the indicated formulism. It may be suspected that the magnetic field intensities systematically somewhat too high may become lower;

3) The problem of the interference with convection by the magnetic field has here received a phenomenological treatment. Although this produced positive results, this situation is not entirely satisfactory. Within the frame of the efforts for a deductive hydrodynamic theory of turbulence, there should also result the basis for a magneto-hydrodynamic theory of turbulence and consequently also a physical explanation for the reduction of the ratio ℓ/H.

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Figure Captions

[Note: "Abb." = "Fig."]

Fig. 1 - Curve of horizontal pressure differential Δp with depth z for $T_{eff} = 4{,}000^{\circ}$ and $\Phi = 5 \times 10^{21}$ Gauss cm^2; the curve family belongs to the different depths z_o of the lower boundary (recognizable at the right end of the curves). The start of the curves at $z \approx 0.7 \times 10^8$ cm corresponds to the geometric depression z_D of the spot.

Fig. 2 - Curve of magnetic field magnitude y and/or mean spot radius $\bar{r}$ with depth z for $T_{eff} = 4{,}000^{\circ}$ and $\Phi = 5 \times 10^{21}$ Gauss cm^2; the curve family belongs to different depths z_o of the lower boundary (recognizable at the right end of the curves). The start of the curves at $z \approx 0.7 \times 10^8$ cm corresponds to the geometric depression z_D of this spot. In part b, the last curve was not entered for the sake of greater clarity.

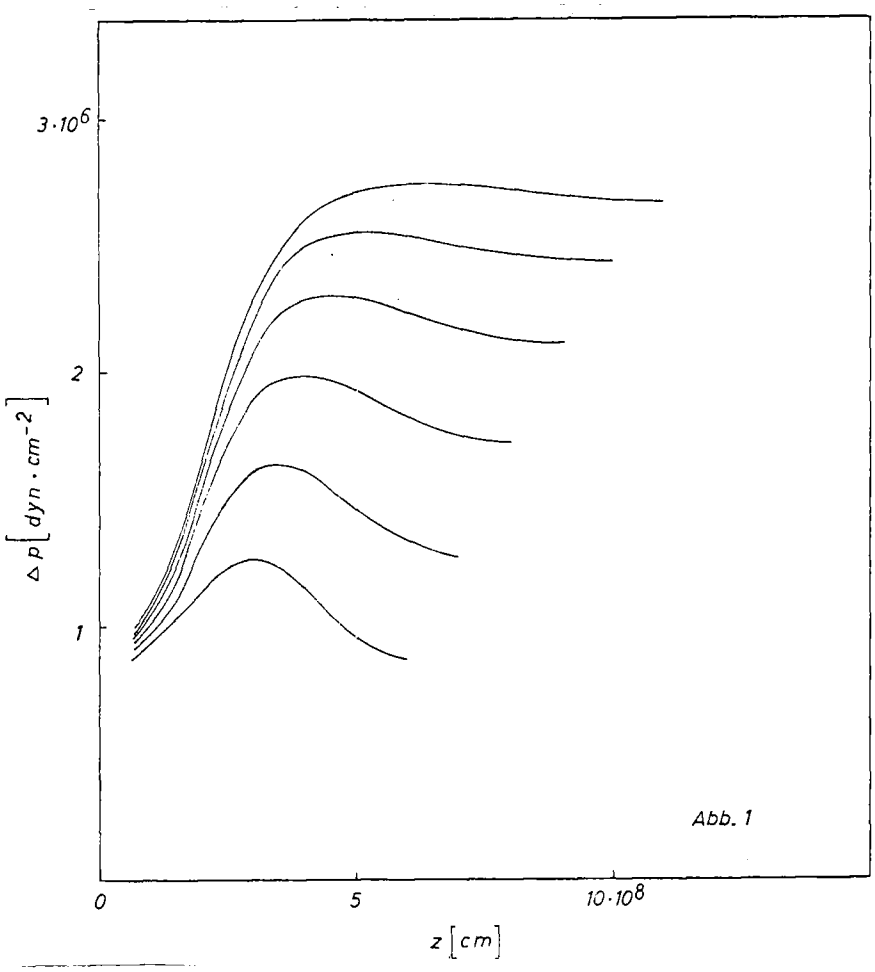
Fig. 3 - Magnetic field magnitude at the upper and/or lower boundary in dependence on the depth z_o of the lower boundary for $\Phi = 5 \times 10^{21}$ Gauss cm^2; the curve family belongs to different effective spot temperatures.

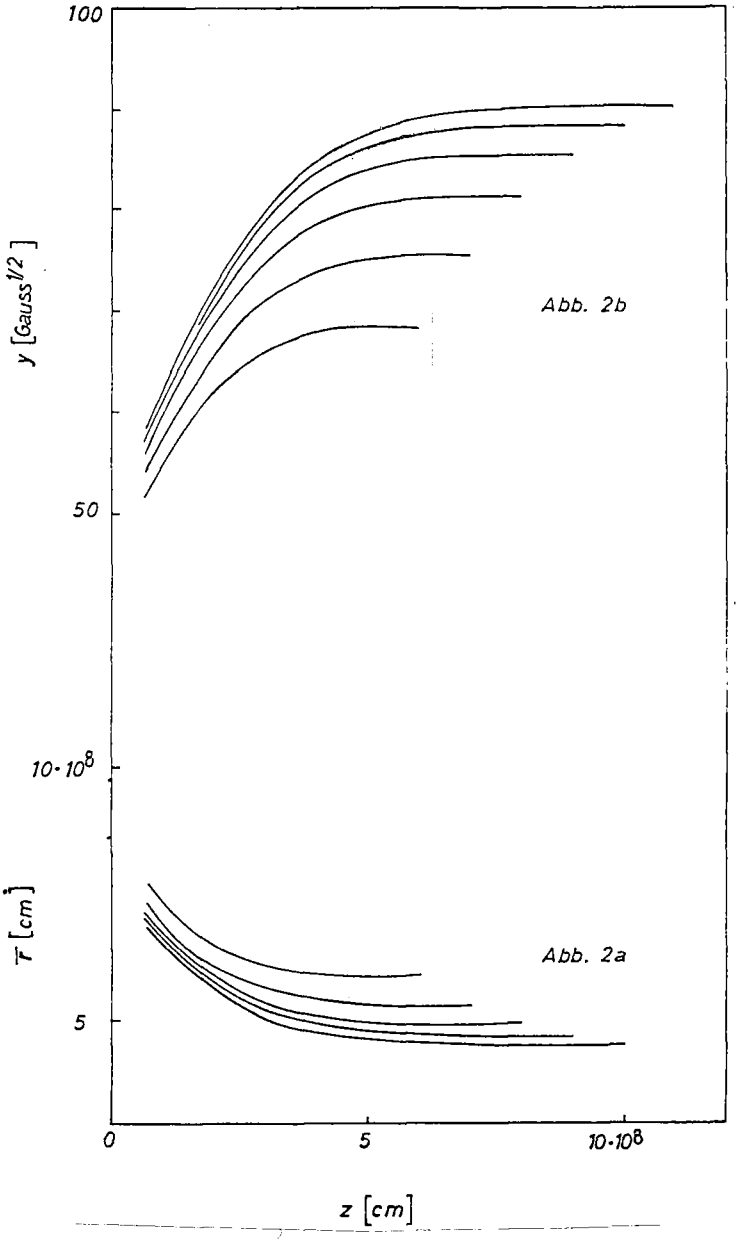
Fig. 4 - Magnetic field magnitude at the upper and/or lower boundary in dependence on the depth z_o of the lower boundary for $T_{eff} = 4{,}000^{\circ}$; the curve family belongs to different magnetic total flux Φ. The scale on the ordinate was doubled as against Fig. 3 for the sake of greater clarity.

Fig. 5 - Relation between magnetic field-center intensity B_c in the observable level and effective spot temperature T_{eff}. The

observations entered for comparison have been taken from P. Stumpff (1961), Fig. 17 and referenced, by means of a relation by J. Houtgast and A. van Sluiters (1948), to magnetic field-center intensities.

Fig. 6 - Relation between magnetic field-center intensity B_c and ratio of mixing length l to homogeneous layer thickness H for demonstration of the interference with convection by the magnetic field. Broken-line parts of the curve are extrapolated.




100
50
10·10^8
5
0
5
10·10^8
y [Gauss^1/2]
r̄ [cm]
z [cm]
Abb. 2b
Abb. 2a

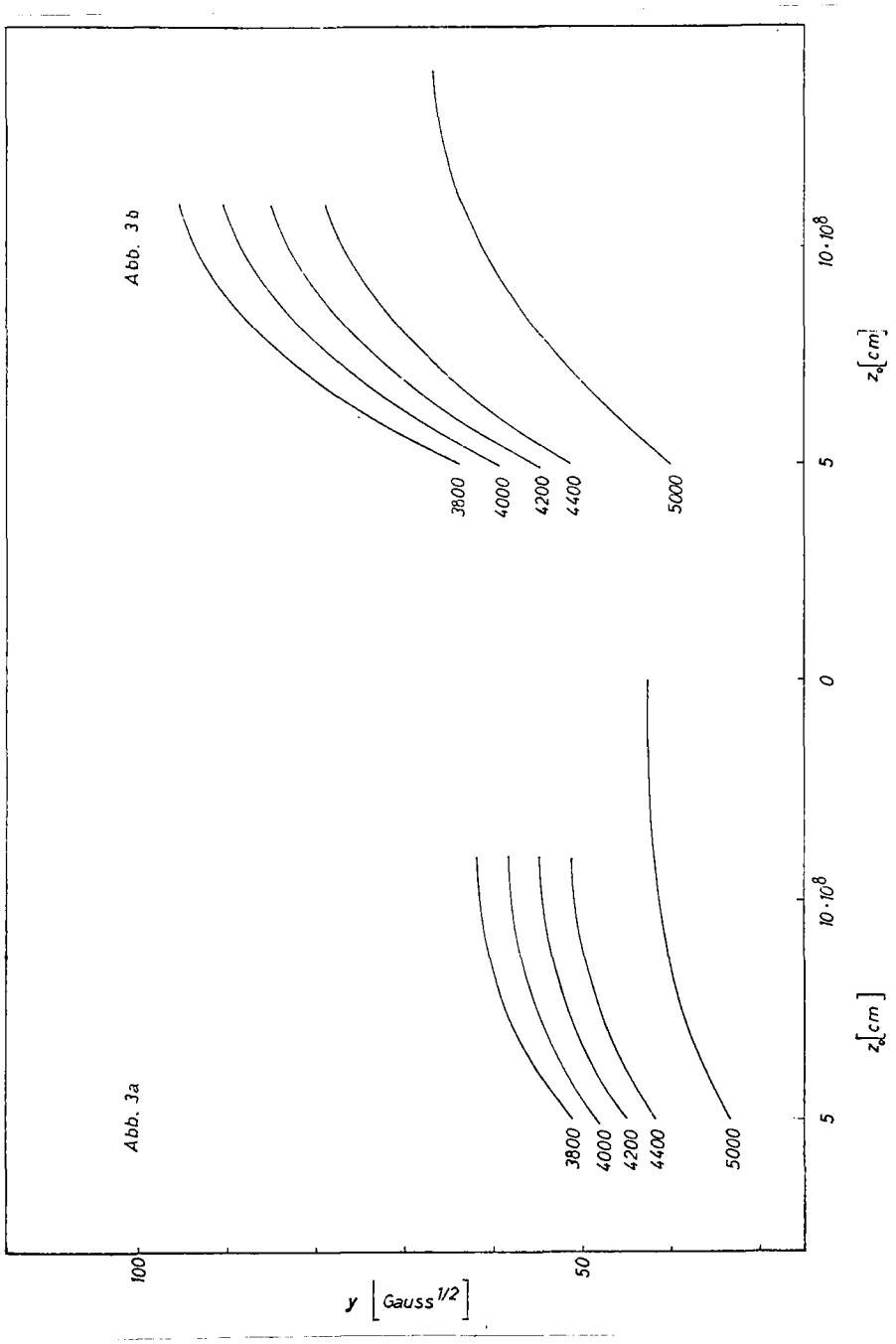


Abb. 3b
3800
4000
4200
4400
5000
Abb. 3a
3800
4000
4200
4400
5000
100
50
y [Gauss^1/2]
5
0
5
10·10^8
10·10^8
z_s[cm]
z_s[cm]

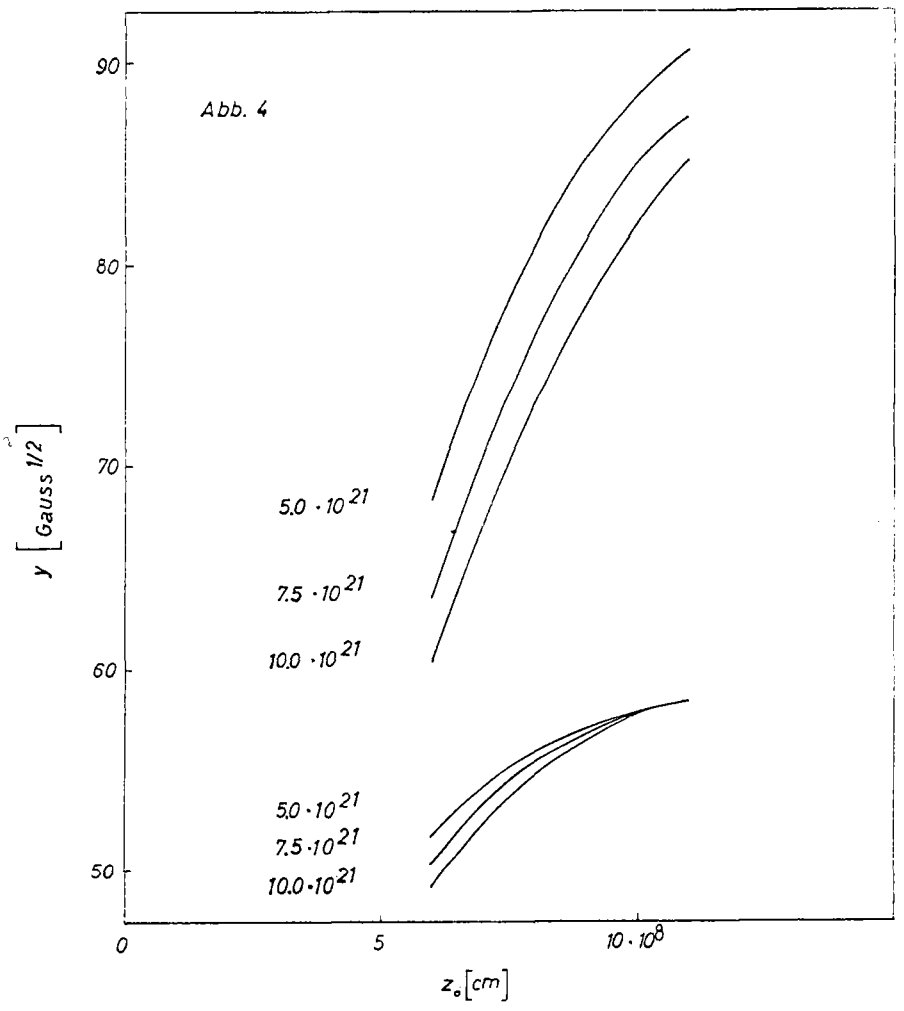


Abb. 4
90
80
70
60
50
y [Gauss^2 1/2]
5.0 · 10^21
7.5 · 10^21
10.0 · 10^21
5.0 · 10^21
7.5 · 10^21
10.0 · 10^21
0
5
10 · 10^8
z_o [cm]

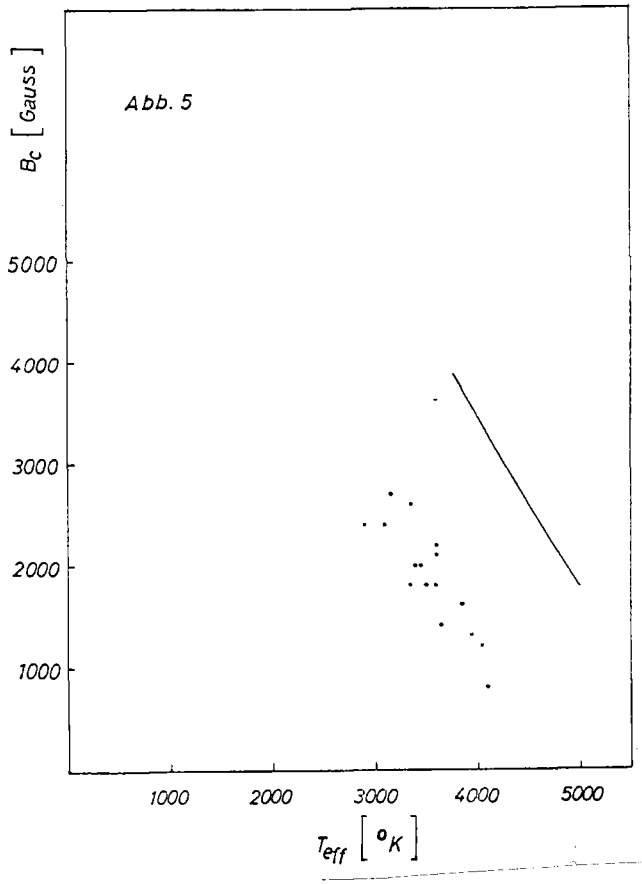


Abb. 5
B_c [Gauss]
5000
4000
3000
2000
1000
1000
2000
3000
4000
5000
T_eff [°K]

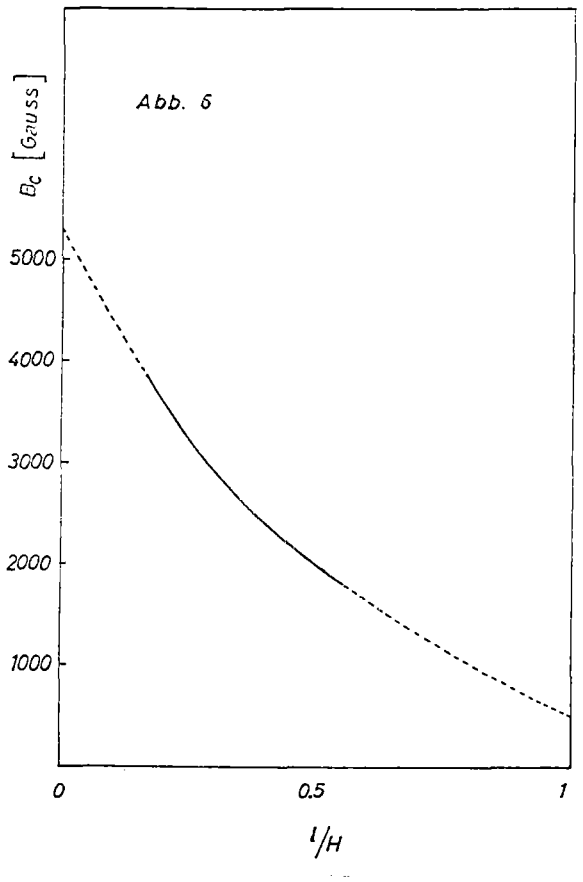


Abb. 6
B_c [Gauss]
5000
4000
3000
2000
1000
0
0.5
1
l/H

Curriculum Vitae:

I was born on 31 Jan. 1935 as son of the municipal administration employee, Ludwig Deiner, and of his wife, Barbara, nee Seller, in Nuremberg. From Sept. 1941, I attended grade school in Nuremberg and subsequently in Sommersdorf near Ausbach (Middle Franconia). After returning to Nuremberg, I entered the "Oberrealschule" on "Woelckerstrasse" in January 1946 and successfully passed the examination for "Matura" in June 1953. With the winter semester, 1953/54, I began the study of physics and mathematics at Erlangen University where I passed the preliminary diploma examination in February 1956 and the final diploma examination in physics in November 1958. My diploma thesis "Analysis of the Light-Curve of the Variable-Cover BV-143" was prepared under the supervision of Dr. E. Mollwo at the Remeis Astronomical Observatory at Bamberg. In January 1959, I joined, as scholarship holder, the Astrophysical Research Team of the Max-Planck Institute for Physics and Astrophysics where I concentrated on the present sunspot investigation.